Non-Parametric Trend Tests and Change-Point Detection

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September 4, 2024

For citation please see `citation(package="trend")`.

Contents

1 Trend detection

1.1 Mann-Kendall Test

The non-parametric Mann-Kendall test is commonly employed to detect monotonic trends in series of environmental data, climate data or hydrological data. The null hypothesis, H_0, is that the data come from a population with independent realizations and are identically distributed. The alternative hypothesis, H_A, is that the data follow a monotonic trend. The Mann-Kendall test statistic is calculated according to :

$$S = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \operatorname{sgn}\left(X_j - X_k\right) \tag{1}$$

with

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases} \tag{2}$$

The mean of S is $E[S] = 0$ and the variance σ^2 is

$$\sigma^2 = \left\{ n\left(n-1\right)\left(2n+5\right) - \sum_{j=1}^{p} t_j\left(t_j-1\right)\left(2t_j+5\right) \right\} / 18 \tag{3}$$

where p is the number of the tied groups in the data set and t_j is the number of data points in the jth tied group. The statistic S is approximately normal distributed provided that the following Z-transformation is employed:

$$Z = \begin{cases} \frac{S-1}{\sigma} & \text{if} \quad S > 0 \\ 0 & \text{if} \quad S = 0 \\ \frac{S+1}{\sigma} & \text{if} \quad S > 0 \end{cases} \tag{4}$$

The statistic S is closely related to Kendall's τ as given by:

$$\tau = \frac{S}{D} \tag{5}$$

where

$$D = \left[\frac{1}{2} n\left(n-1\right) - \frac{1}{2} \sum_{j=1}^{p} t_j\left(t_j-1\right) \right]^{1/2} \left[\frac{1}{2} n\left(n-1\right) \right]^{1/2} \tag{6}$$

The univariate Mann-Kendall test is envoked as folllows:

```
> require(trend)
> data(maxau)
> Q <- maxau[,"Q"]
> mk.test(Q)
```

```
        Mann-Kendall trend test

data:  Q
z = -1.3989, n = 45, p-value = 0.1619
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
 -144.0000000 10450.0000000    -0.1454545
```

1.2 Seasonal Mann-Kendall Test

The Mann-Kendall statistic for the gth season is calculated as:

$$S_g = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \operatorname{sgn}\left(X_{jg} - X_{ig}\right), \quad g = 1, 2, \ldots, m \tag{7}$$

According to Hirsch *et al.* (1982), the seasonal Mann-Kendall statistic, $\hat{S}$, for the entire series is calculated according to

$$\hat{S} = \sum_{g=1}^{m} S_g \tag{8}$$

For further information, the reader is referred to Hipel and McLoed (1994, p. 866-869) and Hirsch *et al.* (1982). The seasonal Mann-Kendall test ist conducted as follows:

```
> require(trend)
> smk.test(nottem)

        Seasonal Mann-Kendall trend test (Hirsch-Slack test)

data:  nottem
z = 2.0919, p-value = 0.03645
alternative hypothesis: true S is not equal to 0
sample estimates:
    S  varS
  224 11364
```

Only the temperature data in Nottingham for August ($S = 80$, $p = 0.009$) as well as for September ($S = 67$, $p = 0.029$) show a significant ($p < 0.05$) positive trend according to the seasonal Mann-Kendall test. Thus, the global trend for the entire series is significant ($S = 224$, $p = 0.036$).

1.3 Correlated Seasonal Mann-Kendall Test

The correlated seasonal Mann-Kendall test can be employed, if the data are coreelated with e.g. the pre-ceeding months. For further information the reader is referred to Hipel and McLoed (1994, p. 869-871).

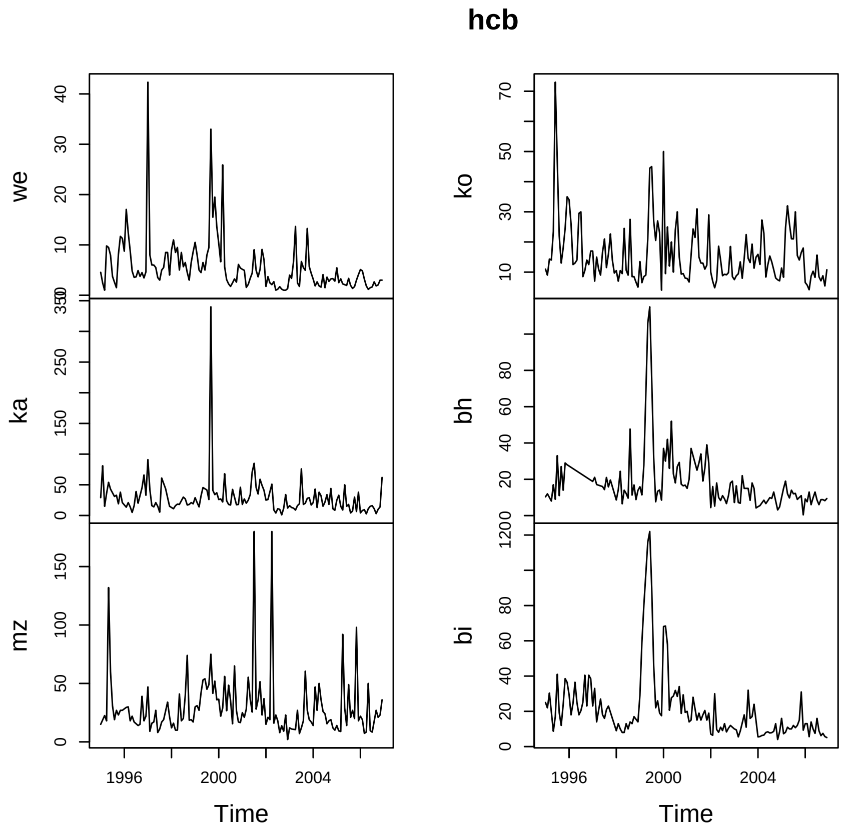
```
> require(trend)
> csmk.test(nottem)

        Correlated Seasonal Mann-Kendall Test

data:  nottem
z = 1.5974, p-value = 0.1102
alternative hypothesis: true S is not equal to 0
sample estimates:
       S     varS
  224.00 19663.33
```

1.4 Multivariate Mann-Kendall Test

Lettenmeier (1988) extended the Mann-Kendall test for trend to a multivariate or multi-site trend test. In this package the formulation of Libiseller and Grimvall (2002) is used for the test.

Particle bound Hexacholorobenzene (HCB, μg kg^{-}1) was monthly measured in suspended matter at six monitoring sites along the river strech of the River Rhine (Pohlert *et al.*, 2011). The below code-snippet tests for trend of each site and for the global trend at the multiple sites.

```
> require(trend)
> data(hcb)
> plot(hcb)
```




```
> ## Single site trends
> site <- c("we", "ka", "mz", "ko", "bh", "bi")
> for (i in 1:6) {print(site[i]) ; print(mk.test(hcb[,site[i]], continuity = TRUE))}
```

```
[1] "we"

        Mann-Kendall trend test

data:  hcb[, site[i]]
z = -5.8753, n = 144, p-value = 4.221e-09
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-3.402000e+03  3.350867e+05 -3.317108e-01

[1] "ka"

        Mann-Kendall trend test
```

```
data:  hcb[, site[i]]
z = -3.5283, n = 144, p-value = 0.0004182
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-2.043000e+03  3.349430e+05 -1.998191e-01

[1] "mz"

        Mann-Kendall trend test

data:  hcb[, site[i]]
z = -1.4447, n = 144, p-value = 0.1485
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-8.370000e+02  3.348423e+05 -8.198541e-02

[1] "ko"

        Mann-Kendall trend test

data:  hcb[, site[i]]
z = -2.7916, n = 144, p-value = 0.005244
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-1.617000e+03  3.350937e+05 -1.575802e-01

[1] "bh"

        Mann-Kendall trend test

data:  hcb[, site[i]]
z = -5.7681, n = 144, p-value = 8.018e-09
alternative hypothesis: true S is not equal to 0
sample estimates:
            S          varS           tau
-3.340000e+03  3.350967e+05 -3.254744e-01

[1] "bi"

        Mann-Kendall trend test
```

```
data:  hcb[, site[i]]
z = -7.1165, n = 144, p-value = 1.107e-12
alternative hypothesis: true S is not equal to 0
sample estimates:
            S           varS            tau
-4.120000e+03  3.350080e+05 -4.023498e-01

> ## Regional trend (all stations including covariance between stations
> mult.mk.test(hcb)

        Multivariate Mann-Kendall Trend Test

data:  hcb
z = -6.686, p-value = 2.293e-11
alternative hypothesis: true S is not equal to 0
sample estimates:
      S    varS
 -15359 5277014
```

1.5 Partial Mann-Kendall Test

This test can be conducted in the presence of co-variates. For full information, the reader is referred to Libiseller and Grimvall (2002).

We assume a correlation between concentration of suspended sediments (s) and flow at Maxau.

```
> data(maxau)
> s <- maxau[,"s"]; Q <- maxau[,"Q"]
> cor.test(s,Q, meth="spearman")

        Spearman's rank correlation rho

data:  s and Q
S = 10564, p-value = 0.0427
alternative hypothesis: true rho is not equal to 0
sample estimates:
      rho
0.3040843
```

As s is significantly positive related to flow, the partial Mann-Kendall test can be employed as follows.

```
> require(trend)
> data(maxau)
> s <- maxau[,"s"]; Q <- maxau[,"Q"]
> partial.mk.test(s,Q)
```

```
        Partial Mann-Kendall Trend Test

data:  t AND  s  .  Q
z = -3.597, p-value = 0.0003218
alternative hypothesis: true S is not equal to 0
sample estimates:
          S          varS           cor
-350.6576077 9503.2897820    0.3009888
```

The test indicates a highly significant decreasing trend ($S = -350.7$, $p < 0.001$) of s, when Q is partialled out.

1.6 Partial correlation trend test

This test performs a partial correlation trend test with either the Pearson's or the Spearman's correlation coefficients (r(tx.z)). The magnitude of the linear / monotonic trend with time is computed while the impact of the co-variate is partialled out.

```
> require(trend)
> data(maxau)
> s <- maxau[,"s"]; Q <- maxau[,"Q"]
> partial.cor.trend.test(s,Q, "spearman")

        Spearman's Partial Correlation Trend Test

data:  t AND  s  .  Q
t = -4.158, df = 43, p-value = 0.0001503
alternative hypothesis: true rho is not equal to 0
sample estimates:
   r(ts.Q)
-0.5355055
```

Likewise to the partial Mann-Kendall test, the partial correlation trend test using Spearman's correlation coefficient indicates a highly significant decreasing trend ($r_{S(ts.Q)} = -0.536$, $n = 45$, $p < 0.001$) of s when Q is partialled out.

1.7 Cox and Stuart Trend Test

The non-parametric Cox and Stuart Trend test tests the first third of the series with the last third for trend.

```
> ## Example from Schoenwiese (1992, p. 114)
> ## Number of frost days in April at Munich from 1957 to 1968
> ## z = -0.5, Accept H0
> frost <- ts(data=c(9,12,4,3,0,4,2,1,4,2,9,7), start=1957)
> cs.test(frost)
```

```
        Cox and Stuart Trend test

data:  frost
z = -0.5, n = 12, p-value = 0.6171
alternative hypothesis: monotonic trend

> ## Example from Sachs (1997, p. 486-487)
> ## z ~ 2.1, Reject H0 on a level of p = 0.0357
> x <- c(5,6,2,3,5,6,4,3,7,8,9,7,5,3,4,7,3,5,6,7,8,9)
> cs.test(x)

        Cox and Stuart Trend test

data:  x
z = 2.0926, n = 22, p-value = 0.03639
alternative hypothesis: monotonic trend
```

2 Magnitude of trend

2.1 Sen's slope

This test computes both the slope (i.e. linear rate of change) and intercept according to Sen's method. First, a set of linear slopes is calculated as follows:

$$d_k = \frac{X_j - X_i}{j - i} \tag{9}$$

for $(1 \leq i < j \leq n)$, where d is the slope, X denotes the variable, n is the number of data, and i, j are indices.

Sen's slope is then calculated as the median from all slopes: $b = \mathrm{Median}\, d_k$. The intercepts are computed for each timestep t as given by

$$a_t = X_t - b * t \tag{10}$$

and the corresponding intercept is as well the median of all intercepts.

This function also computes the upper and lower confidence limits for sens slope.

```
> require(trend)
> s <- maxau[,"s"]
> sens.slope(s)

        Sen's slope

data:  s
z = -3.8445, n = 45, p-value = 0.0001208
alternative hypothesis: true z is not equal to 0
```

```
95 percent confidence interval:
 -0.4196477 -0.1519026
sample estimates:
Sen's slope
 -0.2876139
```

2.2 Seasonal Sen's slope

Acccording to Hirsch *et al.* (1982) the seasonal Sen's slope is calculated as follows:

$$d_{ijk} = \frac{X_{ij} - x_{ik}}{j - k} \tag{11}$$

for each $(x_{ij}, x_{ik}$ pair $i = 1, 2, \ldots, m$, where $1 \leq k < j \leq n_i$ and n_i is the number of known values in the ith season. The seasonal slope estimator is the median of the d_{ijk} values.

```
> require(trend)
> sea.sens.slope(nottem)

[1] 0.05
```

3 Change-point detection

3.1 Pettitt's test

The approach after Pettitt (1979) is commonly applied to detect a single change-point in hydrological series or climate series with continuous data. It tests the H_0: The T variables follow one or more distributions that have the same location parameter (no change), against the alternative: a change point exists. The non-parametric statistic is defined as:

$$K_T = \max |U_{t,T}| , \tag{12}$$

where

$$U_{t,T} = \sum_{i=1}^{t} \sum_{j=t+1}^{T} \operatorname{sgn}(X_i - X_j) \tag{13}$$

The change-point of the series is located at K_T, provided that the statistic is significant. The significance probability of K_T is approximated for $p \leq 0.05$ with

$$p \simeq 2 \, \exp\left(\frac{-6 \, K_T^2}{T^3 + T^2}\right) \tag{14}$$

The Pettitt-test is conducted in such a way:

```
> require(trend)
> data(PagesData)
> pettitt.test(PagesData)

        Pettitt's test for single change-point detection

data:  PagesData
U* = 232, p-value = 0.01456
alternative hypothesis: two.sided
sample estimates:
probable change point at time K
                             17
```

As given in the publication of Pettitt (1979) the change-point of Page's data is located at $t = 17$, with $K_T = 232$ and $p = 0.014$.

3.2 Buishand Range Test

Let X denote a normal random variate, then the following model with a single shift (change-point) can be proposed:

$$x_i = \begin{cases} \mu + \epsilon_i, & i = 1, \ldots, m \\ \mu + \Delta + \epsilon_i & i = m + 1, \ldots, n \end{cases} \tag{15}$$

$\epsilon \approx N(0, \sigma)$. The null hypothesis $\Delta = 0$ is tested against the alternative $\Delta \neq 0$.

In the Buishand range test (Buishand, 1982), the rescaled adjusted partial sums are calculated as

$$S_k = \sum_{i=1}^{k} (x_i - \hat{x}) \qquad (1 \leq i \leq n) \tag{16}$$

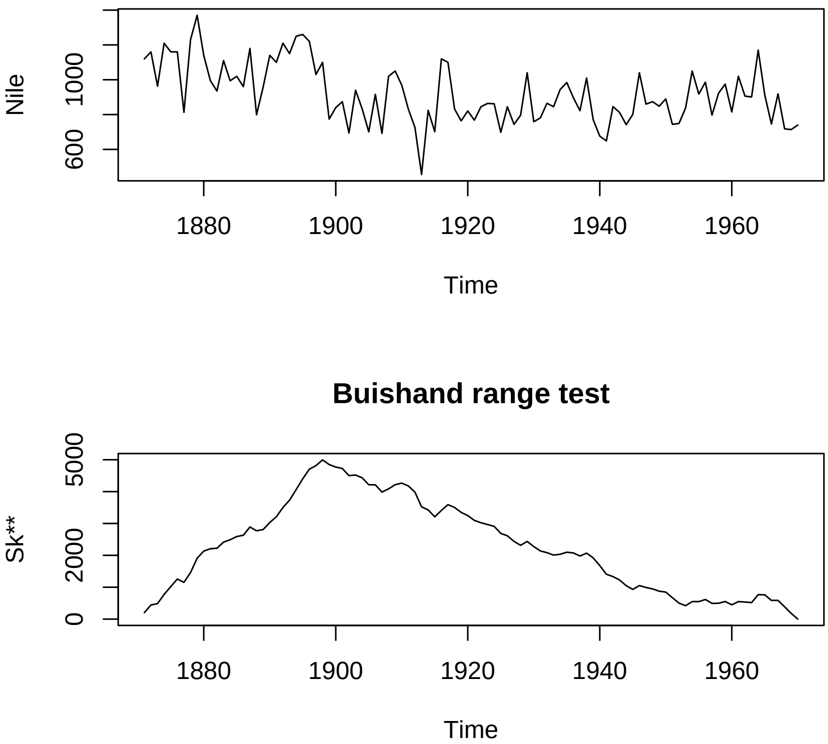
The test statistic is calculated as:

$$Rb = \frac{\max S_k - \min S_k}{\sigma} \tag{17}$$

the p.value is estimated with a Monte Carlo simulation using m replicates.

```
> require(trend)
> (res <- br.test(Nile))

        Buishand range test

data:  Nile
R / sqrt(n) = 2.9518, n = 100, p-value < 2.2e-16
alternative hypothesis: true delta is not equal to 0
sample estimates:
probable change point at time K
                             28
```

```
> par(mfrow=c(2,1))
> plot(Nile); plot(res)
```



3.3 Buishand U Test

In the Buishand U Test (Buishand, 1984), the null hypothesis is the same as in the Buishand Range Test (see Eq. 15). The test statistic is

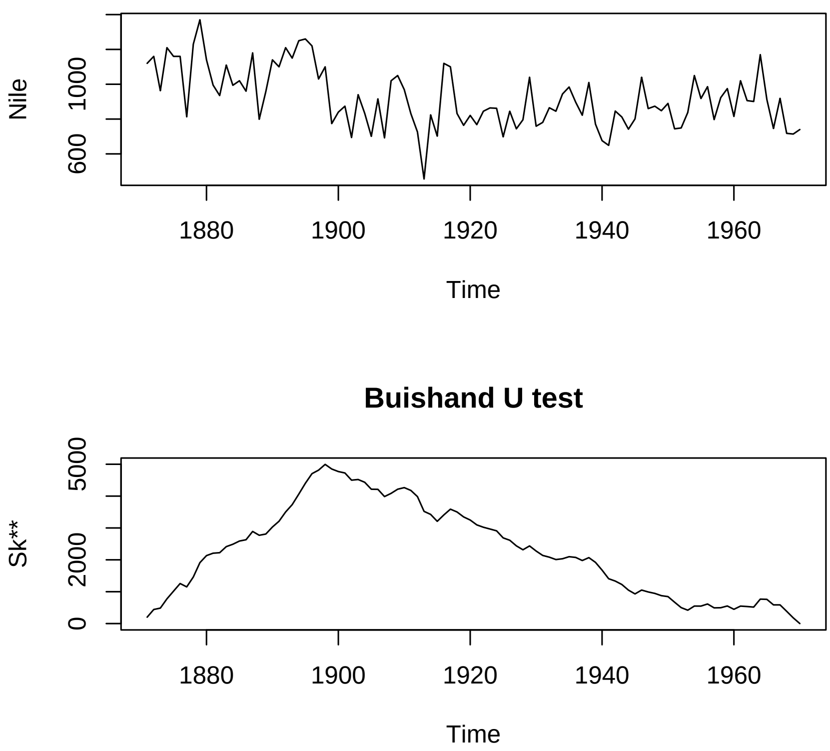
$$U = [n\,(n+1)]^{-1} \sum_{k=1}^{n-1} (S_k / D_x)^2 \tag{18}$$

with

$$D_x = \sqrt{n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})} \tag{19}$$

and S_k as given in Eq. 16. The p.value is estimated with a Monte Carlo simulation using m replicates.

```
> require(trend)
> (res <- bu.test(Nile))

        Buishand U test

data:  Nile
U = 2.4764, n = 100, p-value < 2.2e-16
alternative hypothesis: true delta is not equal to 0
sample estimates:
probable change point at time K
                             28

> par(mfrow=c(2,1))
> plot(Nile); plot(res)
```



3.4 Standard Normal Homogeinity Test

In the Standard Normal Homogeinity Test (**?**), the null hypothesis is the same as in the Buishand Range Test (see Eq. 15). The test statistic is

$$T_k = k z_1^2 + (n-k)\, z_2^2 \qquad (1 \leq k < n) \tag{20}$$

where

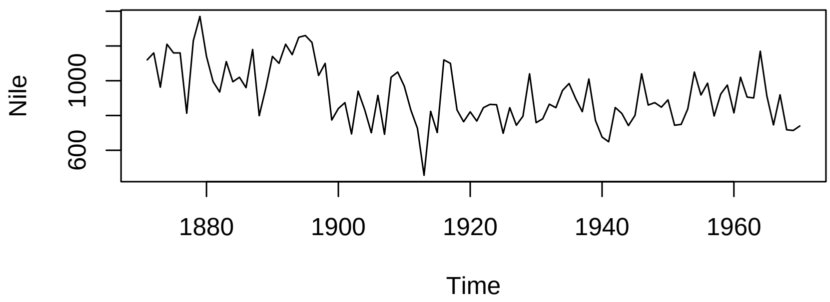
$$z_1 = \frac{1}{k} \sum_{i=1}^{k} \frac{x_i - \bar{x}}{\sigma} \quad z_2 = \frac{1}{n-k} \sum_{i=k+1}^{n} \frac{x_i - \bar{x}}{\sigma}. \tag{21}$$

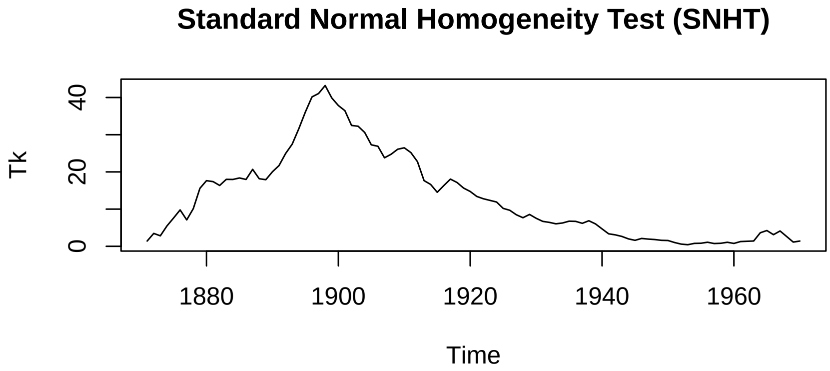
The critical value is:

$$T = \max T_k \tag{22}$$

The p.value is estimated with a Monte Carlo simulation using m replicates.

```
> require(trend)
> (res <- snh.test(Nile))

        Standard Normal Homogeneity Test (SNHT)

data:  Nile
T = 43.219, n = 100, p-value < 2.2e-16
alternative hypothesis: true delta is not equal to 0
sample estimates:
probable change point at time K
                             28

> par(mfrow=c(2,1))
> plot(Nile); plot(res)
```





4 Randomness

4.1 Wallis and Moore phase-frequency test

A phase frequency test was proposed by Wallis and Moore (1941) and is used for testing a series for randomness:

```
> ## Example from Schoenwiese (1992, p. 113)
> ## Number of frost days in April at Munich from 1957 to 1968
> ## z = -0.124, Accept H0
> frost <- ts(data=c(9,12,4,3,0,4,2,1,4,2,9,7), start=1957)
> wm.test(frost)

        Wallis and Moore Phase-Frequency test

data:  frost
z = -0.12384, p-value = 0.9014
alternative hypothesis: The series is significantly different from randomness
```

```
> ## Example from Sachs (1997, p. 486)
> ## z = 2.56, Reject H0 on a level of p < 0.05
> x <- c(5,6,2,3,5,6,4,3,7,8,9,7,5,3,4,7,3,5,6,7,8,9)
> wm.test(x)

        Wallis and Moore Phase-Frequency test

data:  x
z = 2.5513, p-value = 0.01073
alternative hypothesis: The series is significantly different from randomness
```

4.2 Bartels test for randomness

Bartels (1982) has proposed a rank version of von Neumann's ratio test for testing a series for randomness:

```
> ## Example from Schoenwiese (1992, p. 113)
> ## Number of frost days in April at Munich from 1957 to 1968
> ##
> frost <- ts(data=c(9,12,4,3,0,4,2,1,4,2,9,7), start=1957)
> bartels.test(frost)

        Bartels's test for randomness

data:  frost
RVN = 1.3304, p-value = 0.1137
alternative hypothesis: The series is significantly different from randomness

> ## Example from Sachs (1997, p. 486)
> x <- c(5,6,2,3,5,6,4,3,7,8,9,7,5,3,4,7,3,5,6,7,8,9)
> bartels.test(x)

        Bartels's test for randomness

data:  x
RVN = 1.0444, p-value = 0.008371
alternative hypothesis: The series is significantly different from randomness

> ## Example from Bartels (1982, p. 43)
> x <- c(4, 7, 16, 14, 12, 3, 9, 13, 15, 10, 6, 5, 8, 2, 1, 11, 18, 17)
> bartels.test(x)

        Bartels's test for randomness

data:  x
RVN = 0.97626, p-value = 0.009463
alternative hypothesis: The series is significantly different from randomness
```

4.3 Wald-Wolfowitz test for stationarity and independence

Wald and Wolfowitz (1942) have proposed a test for randomness:

```
> ## Example from Schoenwiese (1992, p. 113)
> ## Number of frost days in April at Munich from 1957 to 1968
> ##
> frost <- ts(data=c(9,12,4,3,0,4,2,1,4,2,9,7), start=1957)
> ww.test(frost)

        Wald-Wolfowitz test for independence and stationarity

data:  frost
z = 1.9198, n = 12, p-value = 0.05488
alternative hypothesis: The series is significantly different from
 independence and stationarity

> ## Example from Sachs (1997, p. 486)
> x <- c(5,6,2,3,5,6,4,3,7,8,9,7,5,3,4,7,3,5,6,7,8,9)
> ww.test(x)

        Wald-Wolfowitz test for independence and stationarity

data:  x
z = 2.1394, n = 22, p-value = 0.03241
alternative hypothesis: The series is significantly different from
 independence and stationarity

> ## Example from Bartels (1982, p. 43)
> x <- c(4, 7, 16, 14, 12, 3, 9, 13, 15, 10, 6, 5, 8, 2, 1, 11, 18, 17)
> ww.test(x)

        Wald-Wolfowitz test for independence and stationarity

data:  x
z = 1.7304, n = 18, p-value = 0.08357
alternative hypothesis: The series is significantly different from
 independence and stationarity
```

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