Linear Matrix Inequality Problems

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We consider three distinct linear matrix inequality problems, all written in the form of a dual optimization problem. The first linear matrix inequality problem we will consider is defined by the following optimization equation for some $n \times p$ matrix $\mathbf{B}$ known in advance

$$\begin{array}{lrcl} \underset{\eta,\ \mathbf{Y}}{\text{maximize}} & -\eta & & \\ \text{subject to} & & & \\ & \mathbf{B}\mathbf{Y} + \mathbf{Y}\mathbf{B}^{\intercal} & \preceq & 0 \\ & -\mathbf{Y} & \preceq & -\mathbf{I} \\ & \mathbf{Y} - \eta\mathbf{I} & \preceq & 0 \\ & Y_{11} & = & 1, \quad \mathbf{Y} \in \mathcal{S}^n \end{array}$$

The function `lmi1` takes as input a matrix `B`, and returns the optimal solution using `sqlp`.

```
R> out <- lmi1(B)
```

As a numerical example, consider the following matrix:

```
R> B <- matrix(c(-1,5,1,0,-2,1,0,0,-1), nrow=3)
R> B

     [,1] [,2] [,3]
[1,]   -1    0    0
[2,]    5   -2    0
[3,]    1    1   -1

R> out <- lmi1(B)
```

Here, the output of interest, $\mathbf{P}$, is stored in the vector `y`.

```
R> P <- smat(blk,1, out$y)

             [,1]         [,2]         [,3]
[1,] 1.000000e+00 9.453573e-07 7.251638e-07
[2,] 9.453573e-07 3.244985e+00 1.722086e+00
[3,] 7.251638e-07 1.722086e+00 2.321009e+00
```

The second linear matrix inequality problem is

$$
\begin{array}{lrcl}
\underset{\mathbf{P},\ \mathbf{d}}{\text{maximize}} & -tr(\mathbf{P}) & & \\
\text{subject to} & & & \\
& \mathbf{A}_1\mathbf{P} + \mathbf{P}\mathbf{A}_1{}^{\mathsf{T}} + \mathbf{B} * diag(\mathbf{d}) * \mathbf{B}^{\mathsf{T}} & \preceq & 0 \\
& \mathbf{A}_2\mathbf{P} + \mathbf{P}\mathbf{A}_2{}^{\mathsf{T}} + \mathbf{B} * diag(\mathbf{d}) * \mathbf{B}^{\mathsf{T}} & \preceq & 0 \\
& -\mathbf{d} & \preceq & 0 \\
& \sum_i^p d_i & = & 1
\end{array}
$$

Here, the matrices $\mathbf{B}$, $\mathbf{A}_1$, and $\mathbf{A}_2$ are known in advance.

The function `lmi2` takes the matrices `A1`, `A2`, and `B` as input, and returns the optimal solution using `sqlp`.

```
R> out <- lmi2(A1,A2,B)
```

As a numerical example, consider the following matrices

```
R> A1 <- matrix(c(-1,0,1,0,-2,1,0,0,-1),3,3)

     [,1] [,2] [,3]
[1,]   -1    0    0
[2,]    0   -2    0
[3,]    1    1   -1

R> A2 <- A1 + 0.1*t(A1)

     [,1] [,2] [,3]
[1,] -1.1  0.0  0.1
[2,]  0.0 -2.2  0.1
[3,]  1.0  1.0 -1.1

R> B <- matrix(c(1,3,5,2,4,6),3,2)

     [,1] [,2]
[1,]    1    2
[2,]    3    4
[3,]    5    6

R> out <- lmi2(A1,A2,B)
R> blk <- out$blk
R> At <- out$At
R> C <- out$C
R> b <- out$b

R> out <- sqlp(blk,At,C,b)
```

Like `lmi1`, the outputs of interest $\mathbf{P}$ and $\mathbf{d}$ are stored in the `y` output variable

```
R> n <- ncol(A1)
R> dlen <- ncol(B)
R> N <- n*(n+1)/2

R> P <- smat(blk,1,out$y[1:N])
```

```
          [,1]     [,2]      [,3]
[1,] 1.074734 1.243470  3.575851
[2,] 1.243470 2.366032  6.167900
[3,] 3.575851 6.167900 22.255810

R> d <- out$y[N + c(1:dlen)]

             [,1]
[1,] 1.000000e+00
[2,] 3.355616e-11
```

The final linear matrix inequality problem originates from a problem in control theory ([1]) and requires three matrices be known in advance, **A**, **B**, and **G**

$$
\begin{array}{ll}
\underset{\eta,\,\mathbf{P}}{\text{maximize}} & \eta \\
\text{subject to} & \\
& \begin{bmatrix} \mathbf{AP} + \mathbf{PA}^\mathsf{T} & \mathbf{0} \\ \mathbf{BP} & \mathbf{0} \end{bmatrix} + \eta \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \preceq \begin{bmatrix} -\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}
\end{array}
$$

The function `lmi3` takes as input the matrices `A`, `B`, and `G`, and returns the optimal solution using `sqlp`.

```
R> out <- lmi3(A,B,G)
```

As a numerical example, consider the following matrices

```
R> A <- matrix(c(-1,0,1,0,-2,1,0,0,-1),3,3)

     [,1] [,2] [,3]
[1,]   -1    0    0
[2,]    0   -2    0
[3,]    1    1   -1

R> B <- matrix(c(1,2,3,4,5,6), 2, 3)

     [,1] [,2] [,3]
[1,]    1    3    5
[2,]    2    4    6

R> G <- matrix(1,3,3)

     [,1] [,2] [,3]
[1,]    1    1    1
[2,]    1    1    1
[3,]    1    1    1

R> out <- lmi3(A,B,G)
```

Like the other two linear matrix inequality problems, the matrix of interest is stored in the output vector `y`

```
R> n <- ncol(A)
R> N <- n*(n+1)/2
```

```
R> blktmp <- matrix(list(),1,2)
R> blktmp[[1,1]] <- "s"
R> blktmp[[1,2]] <- n

R> P <- smat(blktmp,1,out$y[1:N])

           [,1]      [,2]       [,3]
[1,]  15.568926 -13.20284  -6.006543
[2,] -13.202839  57.77663 -28.927474
[3,]  -6.006543 -28.92747  39.165821
```

References

[1] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.