Toeplitz Approximation

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Given a symmetric matrix $\mathbf{F}$, the Toeplitz approximation problem seeks to find the nearest symmetric positive definite Toeplitz matrix. In general, a Toeplitz matrix is one with constant descending diagonals, i.e.

$$\mathbf{T} = \begin{bmatrix} a & b & c & d & e \\ f & a & b & c & d \\ g & f & a & b & c \\ h & g & f & a & b \\ i & h & g & f & a \end{bmatrix}$$

is a general Toeplitz matrix. For our specific problem, we seek a *symmetric* Toeplitz matrix, i.e.,

$$\mathbf{T}^* = \begin{bmatrix} a & b & c & d & e \\ b & a & b & c & d \\ c & b & a & b & c \\ d & c & b & a & b \\ e & d & f & b & a \end{bmatrix}$$

The problem is formulated as the following optimization problem

$$\begin{array}{ll} \underset{\mathbf{X}}{\text{maximize}} & -y_{n+1} \\ \text{subject to} & \\ & \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\beta \end{bmatrix} + \sum_{k=1}^{n} y_k \begin{bmatrix} \mathbf{0} & \gamma_k \mathbf{e}_k \\ \gamma_k \mathbf{e}_k^T & -2q_k \end{bmatrix} + y_{n+1}\mathbf{B} \geq \mathbf{0} \\ & [y_1, ..., y_n]^{\mathsf{T}} + y_{n+1}\mathbf{B} \geq \mathbf{0} \end{array}$$

where $\mathbf{B}$ is an $(n+1) \times (n+1)$ matrix of zeros, and $\mathbf{B}_{(n+1)(n+1)} = 1$, $q_1 = -tr(\mathbf{F})$, q_k = sum of k^{th} diagonal upper and lower triangular matrix, $\gamma_1 = \sqrt{n}$, $\gamma_k = \sqrt{2 * (n-k+1)}$, $k = 2, ..., n$, and $\beta = ||\mathbf{F}||_F^2$.

The function `toep` takes as input a symmetric matrix `F` for which we would like to find the nearest Toeplitz matrix, and returns the optimal solution using `sqlp`.

```
R> out <- toep(F)
```

Numerical Example

Consider the following symmetric matrix for which we would like to find the nearest Toeplitz matrix

```
R> data(Ftoep)
```

```
         V1     V2     V3     V4     V5     V6     V7     V8     V9    V10
 [1,]  0.170  0.127  0.652 -0.490  0.963  0.372 -0.707 -0.250 -0.022  1.087
 [2,]  0.127 -1.637  0.031  1.276 -1.475 -1.842 -0.529  1.534 -2.810  0.923
```

```
 [3,]  0.652  0.031  3.339 -0.246  0.249 -2.367  4.327  0.876 -1.832  0.507
 [4,] -0.490  1.276 -0.246 -1.556 -1.415 -0.022 -0.052  1.564 -1.140 -0.982
 [5,]  0.963 -1.475  0.249 -1.415 -0.656 -0.059 -3.101  0.337 -1.526 -0.737
 [6,]  0.372 -1.842 -2.367 -0.022 -0.059  2.617 -0.919  0.869  2.574  0.669
 [7,] -0.707 -0.529  4.327 -0.052 -3.101 -0.919  0.936  1.458 -0.622  1.632
 [8,] -0.250  1.534  0.876  1.564  0.337  0.869  1.458  0.013  1.348  1.736
 [9,] -0.022 -2.810 -1.832 -1.140 -1.526  2.574 -0.622  1.348 -3.817  0.925
[10,]  1.087  0.923  0.507 -0.982 -0.737  0.669  1.632  1.736  0.925  0.527
```

Using `sqlp`, we are interested in the output `Z`, the optimal solution to the dual problem, which will be the nearest symmetric Toeplitz matrix. Note that the final row/column should be removed.

```
R> out <- toep(Ftoep)
```

```
R> F <- out$Z[[1]]
R> F <- F[-nrow(F),]
R> F <- F[,-ncol(F)]
```

```
        [,1]   [,2]   [,3]   [,4]   [,5]   [,6]   [,7]   [,8]   [,9]  [,10]
 [1,]  0.563  0.098 -0.038 -0.113  0.343 -0.054 -0.237 -0.369  0.228  0.077
 [2,]  0.098  0.563  0.098 -0.038 -0.113  0.343 -0.054 -0.237 -0.369  0.228
 [3,] -0.038  0.098  0.563  0.098 -0.038 -0.113  0.343 -0.054 -0.237 -0.369
 [4,] -0.113 -0.038  0.098  0.563  0.098 -0.038 -0.113  0.343 -0.054 -0.237
 [5,]  0.343 -0.113 -0.038  0.098  0.563  0.098 -0.038 -0.113  0.343 -0.054
 [6,] -0.054  0.343 -0.113 -0.038  0.098  0.563  0.098 -0.038 -0.113  0.343
 [7,] -0.237 -0.054  0.343 -0.113 -0.038  0.098  0.563  0.098 -0.038 -0.113
 [8,] -0.369 -0.237 -0.054  0.343 -0.113 -0.038  0.098  0.563  0.098 -0.038
 [9,]  0.228 -0.369 -0.237 -0.054  0.343 -0.113 -0.038  0.098  0.563  0.098
[10,]  0.077  0.228 -0.369 -0.237 -0.054  0.343 -0.113 -0.038  0.098  0.563
```