

Linear Matrix Inequality Problems

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We consider three distinct linear matrix inequality problems, all written in the form of a dual optimization problem. The first linear matrix inequality problem we will consider is defined by the following optimization equation for some $n \times p$ matrix \mathbf{B} known in advance

$$\begin{aligned} & \underset{\eta, \mathbf{Y}}{\text{maximize}} && -\eta \\ & \text{subject to} && \\ & && \mathbf{B}\mathbf{Y} + \mathbf{Y}\mathbf{B}^T \preceq 0 \\ & && -\mathbf{Y} \preceq -\mathbf{I} \\ & && \mathbf{Y} - \eta\mathbf{I} \preceq 0 \\ & && Y_{11} = 1, \quad \mathbf{Y} \in \mathcal{S}^n \end{aligned}$$

The function `lmi1` takes as input a matrix \mathbf{B} , and returns the optimal solution using `sqp1`.

```
R> out <- lmi1(B)
```

As a numerical example, consider the following matrix:

```
R> B <- matrix(c(-1,5,1,0,-2,1,0,0,-1), nrow=3)
R> B
```

```
      [,1] [,2] [,3]
[1,]  -1   0   0
[2,]   5  -2   0
[3,]   1   1  -1
```

```
R> out <- lmi1(B)
```

Here, the output of interest, \mathbf{P} , is stored in the vector `y`.

```
R> P <- smat(blk,1, out$y)
```

```
      [,1]      [,2]      [,3]
[1,] 1.000000e+00 9.453573e-07 7.251638e-07
[2,] 9.453573e-07 3.244985e+00 1.722086e+00
[3,] 7.251638e-07 1.722086e+00 2.321009e+00
```

The second linear matrix inequality problem is

$$\begin{aligned}
& \underset{\mathbf{P}, \mathbf{d}}{\text{maximize}} && -\text{tr}(\mathbf{P}) \\
& \text{subject to} && \\
& && \mathbf{A}_1\mathbf{P} + \mathbf{P}\mathbf{A}_1^\top + \mathbf{B} * \text{diag}(\mathbf{d}) * \mathbf{B}^\top \preceq 0 \\
& && \mathbf{A}_2\mathbf{P} + \mathbf{P}\mathbf{A}_2^\top + \mathbf{B} * \text{diag}(\mathbf{d}) * \mathbf{B}^\top \preceq 0 \\
& && -\mathbf{d} \preceq 0 \\
& && \sum_i^p d_i = 1
\end{aligned}$$

Here, the matrices \mathbf{B} , \mathbf{A}_1 , and \mathbf{A}_2 are known in advance.

The function `lmi2` takes the matrices `A1`, `A2`, and `B` as input, and returns the optimal solution using `sqp`.

```
R> out <- lmi2(A1,A2,B)
```

As a numerical example, consider the following matrices

```
R> A1 <- matrix(c(-1,0,1,0,-2,1,0,0,-1),3,3)
```

```
      [,1] [,2] [,3]
[1,]  -1   0   0
[2,]   0  -2   0
[3,]   1   1  -1
```

```
R> A2 <- A1 + 0.1*t(A1)
```

```
      [,1] [,2] [,3]
[1,] -1.1  0.0  0.1
[2,]  0.0 -2.2  0.1
[3,]  1.0  1.0 -1.1
```

```
R> B <- matrix(c(1,3,5,2,4,6),3,2)
```

```
      [,1] [,2]
[1,]   1   2
[2,]   3   4
[3,]   5   6
```

```
R> out <- lmi2(A1,A2,B)
```

```
R> blk <- out$blk
```

```
R> At <- out$At
```

```
R> C <- out$C
```

```
R> b <- out$b
```

```
R> out <- sqlp(blk,At,C,b)
```

Like `lmi1`, the outputs of interest \mathbf{P} and \mathbf{d} are stored in the `y` output variable

```
R> n <- ncol(A1)
```

```
R> dlen <- ncol(B)
```

```
R> N <- n*(n+1)/2
```

```
R> P <- smat(blk,1,out$y[1:N])
```

```

      [,1] [,2] [,3]
[1,] 1.074734 1.243470 3.575851
[2,] 1.243470 2.366032 6.167900
[3,] 3.575851 6.167900 22.255810

```

```
R> d <- out$y[N + c(1:dlen)]
```

```

      [,1]
[1,] 1.000000e+00
[2,] 3.355616e-11

```

The final linear matrix inequality problem originates from a problem in control theory ([1]) and requires three matrices be known in advance, \mathbf{A} , \mathbf{B} , and \mathbf{G}

$$\begin{array}{l}
 \text{maximize } \eta \\
 \eta, \mathbf{P} \\
 \text{subject to} \\
 \left[\begin{array}{cc} \mathbf{AP} + \mathbf{PA}^\top & \mathbf{0} \\ \mathbf{BP} & \mathbf{0} \end{array} \right] + \eta \left[\begin{array}{cc} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{array} \right] \preceq \left[\begin{array}{cc} -\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{array} \right]
 \end{array}$$

The function `lmi3` takes as input the matrices \mathbf{A} , \mathbf{B} , and \mathbf{G} , and returns the optimal solution using `sqp`.

```
R> out <- lmi3(A,B,G)
```

As a numerical example, consider the following matrices

```
R> A <- matrix(c(-1,0,1,0,-2,1,0,0,-1),3,3)
```

```

      [,1] [,2] [,3]
[1,] -1    0    0
[2,]  0   -2    0
[3,]  1    1   -1

```

```
R> B <- matrix(c(1,2,3,4,5,6), 2, 3)
```

```

      [,1] [,2] [,3]
[1,]  1    3    5
[2,]  2    4    6

```

```
R> G <- matrix(1,3,3)
```

```

      [,1] [,2] [,3]
[1,]  1    1    1
[2,]  1    1    1
[3,]  1    1    1

```

```
R> out <- lmi3(A,B,G)
```

Like the other two linear matrix inequality problems, the matrix of interest is stored in the output vector `y`

```

R> n <- ncol(A)
R> N <- n*(n+1)/2

```

```
R> blktmp <- matrix(list(),1,2)
R> blktmp[[1,1]] <- "s"
R> blktmp[[1,2]] <- n

R> P <- smat(blktmp,1,out$y[1:N])

      [,1]      [,2]      [,3]
[1,] 15.568926 -13.20284 -6.006543
[2,] -13.202839  57.77663 -28.927474
[3,] -6.006543 -28.92747  39.165821
```

References

- [1] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.