Testing the Ratio of Two Poisson Rates

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November 16, 2024

1 Example

Here is a quick example of the function `rateratio.test`. Suppose you have two rates that you assume are Poisson and you want to test that they are different. Suppose you observe 2 events with time at risk of $n = 17877$ in one group and 9 events with time at risk of $m = 16660$ in another group. Here is the test:

```
> rateratio.test(c(2,9),c(n,m))

        Exact Rate Ratio Test, assuming Poisson counts

data:  c(2, 9) with time of c(n, m), null rate ratio 1
p-value = 0.05011
alternative hypothesis: true rate ratio is not equal to 1
95 percent confidence interval:
 0.02177406 1.00054910
sample estimates:
  Rate Ratio       Rate 1       Rate 2 
0.2070941557 0.0001118756 0.0005402161 
```

The result is barely non-significant at the 0.05 level. This example was chosen to make a point, that is why the p-value is so close to 0.05. See Section 5 below.

2 Assumptions and Notation

Assume that $Y \sim Poisson(n\lambda_y)$ and $X \sim Poisson(m\lambda_x)$. We are interested in the rate ratio, $\theta = \lambda_y/\lambda_x$. The parameters n and m are assumed known and represent the time spent in the Poisson process with the given rates. For example, n could the the number of person-years at risk associated with Y. We wish to test one of the three following hypotheses:

less

$$\begin{aligned} H_0: &\quad \theta \geq \Delta \\ H_1: &\quad \theta < \Delta \end{aligned}$$

greater

$$\begin{aligned} H_0: &\quad \theta \leq \Delta \\ H_1: &\quad \theta > \Delta \end{aligned}$$

two-sided

$$\begin{aligned} H_0: &\quad \theta = \Delta \\ H_1: &\quad \theta \neq \Delta \end{aligned}$$

For the tests using the rate ratios, we can use the uniformly most powerful (UMP) unbiased test. This test is based on conditioning on the sum $X + Y$ (see e.g., Lehmann and Romano, 2005, p. 125 or p. 152 of Lehmann, 1986). We modify Lehmann's presentation by allowing the constants m and n, representing the time in the Poisson process. We have that

$$Y|X + Y = t \sim Binomial\,(t, p(\theta))$$

where

$$p(\theta) = \frac{n\lambda_y}{n\lambda_y + m\lambda_x} = \frac{n\theta}{n\theta + m}. \tag{1}$$

3 Confidence Intervals

Since $p(\theta)$ is a monotonic increasing function of θ, if we have exact confidence intervals for $p(\theta)$, then we can transform them to exact confidence intervals for θ. The R function binom.test gives exact intervals for binomial observations (see Clopper and Pearson, 1934 or Leemis and Trivedi, 1996). We write the $100(1-\alpha)\%$ one-sided lower confidence limit for p as $L_p(Y;\alpha)$ and the $100(1-\alpha)\%$ one-sided upper confidence limit for p as $U_p(Y;\alpha)$. For the $100(1-\alpha)\%$ two-sided cofidence interval, binom.test and Clopper and Pearson (1934) use the central confidence interval defined as $[L_p(Y;\alpha/2), L_p(Y;\alpha/2)]$. The *central* confidence interval guarantees that

$$Pr[p < L_p(Y;\alpha/2)|p,t] \leq \alpha/2 \text{ for all } p \text{ and } t$$

and

$$Pr[p > U_p(Y;\alpha/2)|p,t] \leq \alpha/2 \text{ for all } p \text{ and } t$$

For shorter exact intervals which are not central see Blaker (2000) and the references therein.

To obtain confidence intervals for θ we set

$$L_p(Y;\alpha) = \frac{nL_\theta(Y;\alpha)}{nL_\theta(Y;\alpha) + m},$$

and perform some algebra to get

$$L_\theta(Y;\alpha) = \frac{mL_p(Y;\alpha)}{n\{1 - L_p(Y;\alpha)\}}.$$

Similarly,

$$U_\theta(Y;\alpha) = \frac{mU_p(Y;\alpha)}{n\{1 - U_p(Y;\alpha)\}}.$$

4 P-values

Just as in the last section, we can use results from the tests of p and translate them to tests of θ. Thus, for example the one-sided p-value of the test with the alternative hypothesis that $\theta > \Delta$ is equivalent to the one-sided p-value of the test that $p > p(\Delta)$. For the two-sided p-value we use the minimum of 1 or twice the minimum of the two one-sided p-values. There are other ways to define the two-sided p-value but they do not give equivalent inferences with the confidence intervals described above (see Section 5 below).

5 Relationship to Other Tests

In the R function `binom.test` (as least up until R version 4.4.2 (2024-10-31)) the two-sided p-value is calculated by defining more extreme responses as those values with binomial density functions less than or equal to the observed density. This is a valid and reasonable way of defining two-sided p-values but it *does not match* with the two-sided confidence intervals. Returning to our example from Section 1 but using `binom.test` we can match the confidence intervals by using equation 1.

```
> n<-17877
> m<-16674
> rateratio.test(c(2,9),c(n,m))$conf.int

[1] 0.02179236 1.00138990
attr(,"conf.level")
[1] 0.95

> b.ci<-binom.test(2,2+9,p=n/(n+m))$conf.int
> theta.ci<-m*b.ci/(n*(1-b.ci))
> theta.ci

[1] 0.02179236 1.00138990
attr(,"conf.level")
[1] 0.95
```

However, the p-values do not match for a two-sided test of $p(1) = n/(n+m)$.

```
> R.Version()$version.string

[1] "R version 4.4.2 (2024-10-31)"

> rateratio.test(c(2,9),c(n,m))

        Exact Rate Ratio Test, assuming Poisson counts

data:  c(2, 9) with time of c(n, m), null rate ratio 1
p-value = 0.05027
```

```
alternative hypothesis: true rate ratio is not equal to 1
95 percent confidence interval:
 0.02179236 1.00138990
sample estimates:
  Rate Ratio       Rate 1       Rate 2
0.2072681844 0.0001118756 0.0005397625

> binom.test(2,2+9,p=n/(n+m))

        Exact binomial test

data:  2 and 2 + 9
number of successes = 2, number of trials = 11, p-value = 0.03315
alternative hypothesis: true probability of success is not equal to 0.517409
95 percent confidence interval:
 0.0228312 0.5177559
sample estimates:
probability of success
             0.1818182
```

The p-values for `rateratio.test` are internally consistent, i.e., if the two-sided p-value is less than α then the $100(1-\alpha/2)\%$ confidence interval does not contain Δ. In contrast the p-values for `binom.test` are not internally consistent as shown by the example. A similar internal inconsistency happens with `fisher.test`.

```
> fisher.test(matrix(c(2,9,n-2,m-9),2,2))

        Fisher's Exact Test for Count Data

data:  matrix(c(2, 9, n - 2, m - 9), 2, 2)
p-value = 0.03312
alternative hypothesis: true odds ratio is not equal to 1
95 percent confidence interval:
 0.02178155 1.00121047
sample estimates:
odds ratio
 0.2072015
```

References

Blaker, H. (2000). "Confidence curves and improved exact confidence intervals for discrete distributions" *Canadian Journal of Statistics* **28,** 783-798 (correction **29,** 681).

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